$\Phi_i(t)$ = potential of conductor ***i*** at time t

$\Phi_x(t)$ = potential of point X at time t

Note that $d_{ii} = a_i$, the radius of conductor ***i***.

Equation (3.17) expresses the potential difference between conductor ***i*** and an arbitrarily selected point x. If point x is taken to infinity, the voltage V_{ix} will become the absolute voltage of conductor i, V_i. To derive the absolute voltage of conductor i, the general expression for V_i is rewritten as:

$$V_{ix}(t) = \frac{1}{2\pi\varepsilon}\sum_{j=1}^{n} q_j(t)\ln\frac{1}{d_{ij}} - \frac{1}{2\pi\varepsilon}\sum_{j=1}^{n} q_j(t)\ln\frac{1}{d_{jx}}$$

Now observe that if the n conductors are the only objects with electric charge, the sum of the electric charges, $q_1(t),\ldots,\ q_n(t)$, must equal zero, that is,

$$\sum_{j=1}^{n} q_j(t) = 0 \tag{3.18}$$

In this case it can be shown that (the reader is encouraged to prove it):

$$\lim_{x\to\infty}\frac{1}{2\pi\varepsilon}\sum_{j=1}^{n} q_j(t)\ln\frac{1}{d_{jx}} = 0 \tag{3.19}$$

Then the absolute voltage of conductor i is

$$V_i(t) = \frac{1}{2\pi\varepsilon}\sum_{j=1}^{n} q_j(t)\ln\frac{1}{d_{ij}} \tag{3.20}$$

The proof of the limit of Eq. (3.19) follows.

Proof: Equation (3.18) is solved for $q_n(t)$:

$$q_n(t) = -\sum_{j=1}^{n-1} q_j(t)$$

Then substitute above in the quantity of equation (3.19) and rearrange to obtain.

$$\frac{1}{2\pi\varepsilon}\sum_{j=1}^{n} q_j(t)\ln\frac{1}{d_{jx}} = \frac{1}{2\pi\varepsilon}\sum_{j=1}^{n-1} q_j(t)\ln\frac{1}{d_{jx}} + \frac{1}{2\pi\varepsilon}q_n(t)\ln\frac{1}{d_{nx}} = \frac{1}{2\pi\varepsilon}\sum_{j=1}^{n-1} q_j(t)\ln\frac{d_{nx}}{d_{jx}}$$

Note that as $x \to \infty$, the ratio $\frac{d_{nx}}{d_{jx}} \to 1.0$. The logarithm of this ratio goes to zero. Thus equation (3.19).

It is expedient to repeat the assumptions under which Eq. (3.20) has been obtained:

Assumption 1: The sum of all charges equals zero, i.e. $\sum_{j=1}^{n} q_j(t) = 0$, and **Assumption 2**: The electric charge is uniformly distributed on the surface of the conductors. This assumption is equivalent to: $d_{ij} \gg a_i, \quad i \neq j$.

The first assumption is valid for any transmission line configuration, assuming that all conductors have been accounted for. For overhead lines, since the conducting soil represents one of the conductors, this means that the earth must be also accounted for. This issue is addressed in the next section. The second assumption is always valid for overhead circuits. For circuits with bundled conductors, three phase cables, etc. the assumption may not result in accurate results. More sophisticated computational methods must be employed in these cases.

Equation (3.20) can be transformed into an equation relating the conductor capacitive current to the conductor voltage. For this purpose, Eq. (3.20) is differentiated with respect to time, yielding.

$$\frac{dv_i(t)}{dt} = \sum_{j=1}^{n} \frac{1}{2\pi\varepsilon} \ln\left(\frac{1}{d_{ij}}\right) \frac{dq_j(t)}{dt}$$

By definition, the time derivative of the conductor electric charge is the capacitive current of the conductor (or charging current):

$$\frac{dq_j(t)}{dt} = i_j^{'}(t) \quad \text{capacitive current of conductor j}$$

Upon substitution, we have

$$\frac{dv_i(t)}{dt} = \sum_{j=1}^{n} \frac{1}{2\pi\varepsilon} \ln\left(\frac{1}{d_{ij}}\right) i_j^{'}(t) \tag{3.21}$$

Equation (3.21) is the basic equation for modeling the capacitive effects of a multiconductor power line. For sinusoidal steady-state analysis, Eq. (3.21) is converted into an algebraic equation. For this purpose, recall that under sinusoidal steady-state conditions, the voltage and currents will have the following general time variation:

$$v_i(t) = \mathrm{Re}\left[\sqrt{2}\tilde{V}_i e^{j\omega t}\right]$$

$$i_i^{'}(t) = \mathrm{Re}\left[\sqrt{2}\tilde{I}_i^{'} e^{j\omega t}\right]$$

where $\tilde{V}_i$ *and* $\tilde{I}_i^{'}$ are complex numbers representing the phasors of the voltage and the capacitive current. Substitution in Eq. (3.21) and solution for $\tilde{V}_i$ gives us

$$\tilde{V}_i = \sum_{j=1}^{n} \frac{1}{j\omega 2\pi\varepsilon} \ln\left(\frac{1}{d_{ij}}\right) \tilde{I}_j^{'}, \quad i = 1, 2, ..., n \qquad (3.22)$$

where $d_{ii} = a_i$, the radius of the conductor i.

It is expedient to define the following quantities:

$$x'_{ij} = \frac{1}{j\omega 2\pi\varepsilon} \ln\frac{1}{d_{ij}} \quad i \neq j \ \text{ ohm·meters} \qquad (3.23a)$$

$$x'_{ii} = \frac{1}{j\omega 2\pi\varepsilon} \ln\frac{1}{a_i} \ \text{ ohm·meters} \qquad (3.23b)$$

which will be called the <u>separation component of the capacitive reactance</u> and the <u>conductor component of the capacitive reactance</u> respectively. These quantities depend on the geometry and material of the components of the capacitive reactance. Using these variables, Equation (3.22) takes the following simple form:

$$\tilde{V}_i = \sum_{j=1}^{n} x_{ij} \tilde{I}_i^{'} \qquad (3.24)$$

It is noted that the components of capacitive reactance for all commercially available conductors have been tabulated. As in the case of the components of inductive reactance, note that the mathematically rigorous reader will be offended by the expressions for x'_{ii} and x'_{ij} since they involve the terms $\ln\frac{1}{a_i}$ and $\ln\frac{1}{d_{ij}}$. It should be observed that if the quantities a_i and d_{ij} are expressed in the same units, the final result will be correct. For this reason it has been accepted that a_i and d_{ij} will be expressed in feet under the understanding that each quantity x'_{ii}, x'_{ij} is meaningless if considered individually.

In summary, the capacitive effects of a power line are represented with Eq. (3.21). Specifically, for each conductor in a power line, one equation can be written relating the capacitive current of the conductors and the time derivative of the conductor voltage. For sinusoidal steady-state analysis, these equations are converted into a set of algebraic equations [Equation (3.24)] relating the phasors of the conductor capacitive currents to the phasor of the conductor voltage.

3.2.3.3 Capacitive Equations of a Multi-Conductor Line Above Earth

Most overhead transmission lines have ground wires to protect them against lightning. Overhead distribution lines have neutral conductors for unbalanced current return. All overhead power lines are suspended above earth. Neutral/ground wires and the earth are conducting media in the vicinity of the line which may be charged with electric charge due to the electric field of the line. Alternatively, these conducting media alter the electric field of the line and affect the capacitance of the line. In this section we examine methods by which the effects of earth and neutral or overhead ground wires on line capacitance can be quantified.

The effect of neutral/ground wires can be computed in a straightforward way by treating these wires in the same way as the phase conductors. It should be observed that the voltage of the neutral/ground wires will be much different from the voltage of the phase conductors. Actually, the voltage of neutral or ground wires is approximately zero at normal operating conditions. For usual applications it is assumed that the voltage of neutral or ground wires is exactly zero.

Computation of the effect of earth on the capacitive reactance of a line, in general, is a difficult problem. To simplify the analysis, it is assumed that the earth is a semi-infinite perfectly conducting medium. In this case the theory of images is applied directly, yielding a rather simple analysis procedure. Specifically, the problem of a transmission line located above earth is replaced with another equivalent problem which does not include the earth, but includes the images of the conductors with respect to the surface of the earth.

Consider a multi-conductor line above earth. The space around the line consists of two media: a non-conducting and a highly conducting medium. Assume the interface to be a plane, as illustrated in Figure 3.19. The conductors of the line are located in the non-conducting medium. Earth conductor is charged with electric charge. The charged conductors will establish an electric field in medium 1. The electric field in medium 2 will be zero since medium 2 is a perfect conductor. The theory of images [???] guarantees that the electric field in the space of medium 1 established by the charged conductor is identical to the electric field generated by two conductors, the original conductor located in the free space, and another which is the geometric image of the actual conductor with respect to the plane interface of the two media. If the electric charge on the conductor is q, the electric charge of its image is -q. This condition guarantees that the electric field intensity on the interface will be perpendicular to the plane interface. Thus the boundary conditions of the problem are matched. A consequence of this condition is that if the voltage of the conductor is V, the voltage of its image will be -V.

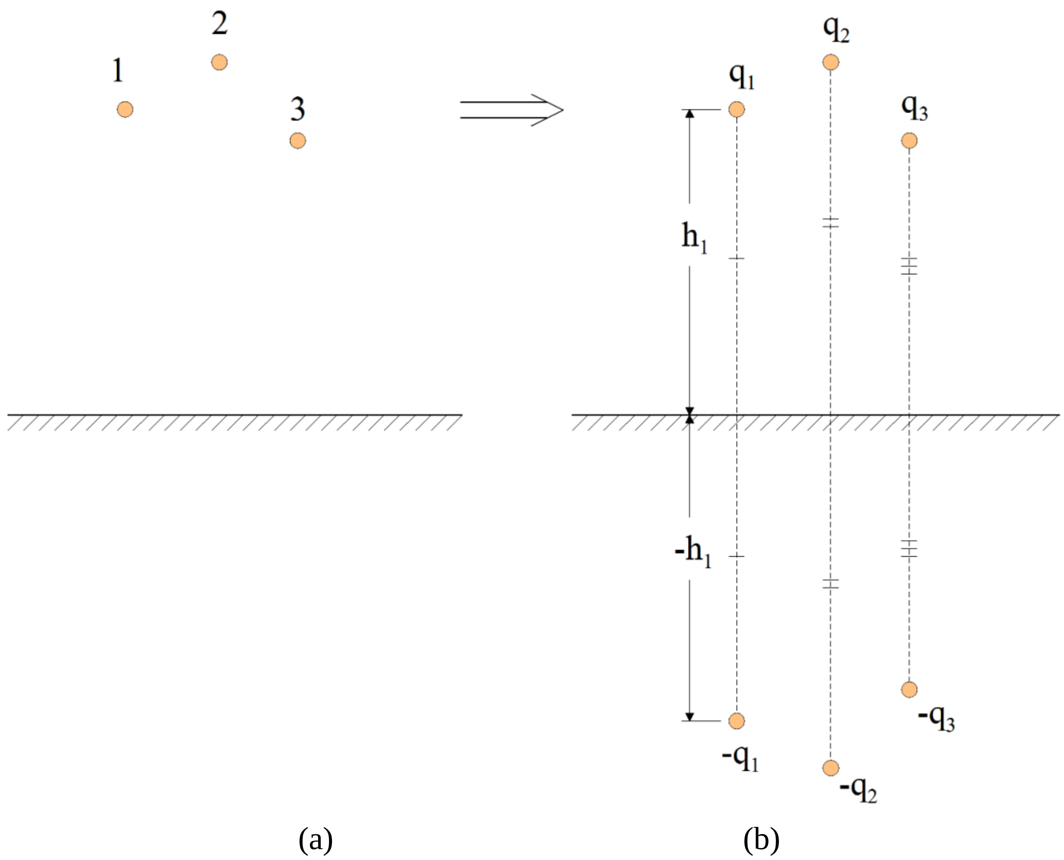


Figure 3.19: Multi-conductor Line Above Earth
[(a) Conductor Arrangement, (b) Conductor and Image Arrangement]

Consider the general transmission line suspended above earth, as illustrated in Figure 3.19a. Application of the theory of images results in the equivalent configuration of Figure 3.19b. Subsequently, the capacitive currents of the conductors are computed as follows: The voltages of the conductors, $\tilde{V}_1, \tilde{V}_2, ..., \tilde{V}_n$ are expressed in terms of the capacitive currents $\tilde{I}_1, \tilde{I}_2, ..., \tilde{I}_n$. In this analysis the capacitive currents of the images are also included. The voltage of conductor i will be:

$$\tilde{V}_i = \sum_{j=1}^{n} x'_{ij} \tilde{I}'_j - \sum_{j=1}^{n} x'_{ij'} \tilde{I}'_j, \quad i = 1, 2, ..., n \tag{3.26}$$

where:

$$x'_{ij} = \frac{1}{j\omega 2\pi\varepsilon} \ln \frac{1}{d_{ij}}$$

$$x'_{ij'} = \frac{1}{j\omega 2\pi\varepsilon} \ln \frac{1}{d_{ij'}}$$

d_{ij} = distance between conductors i, j

$d_{ij'}$ = distance between conductors i, and the image of conductor j (which is the same as the distance between conductor j and the image of conductor i)

Equation (3.26) is rewritten by combining the terms with the same electric current, yielding the compact form:

$$\tilde{V}_i = \sum_{j=1}^{n} \frac{1}{j\omega 2\pi\varepsilon} \ln\left(\frac{d_{ij'}}{d_{ij}}\right) \tilde{I}_j^{'}, \quad i = 1, 2, \ldots, n \tag{3.27}$$

Assuming that the voltages $\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_n$ are known, Eq. (3.27) is solved to provide the capacitive currents $\tilde{I}_1, \tilde{I}_2, \ldots, \tilde{I}_n$. The earth will also carry a capacitive currents, $\tilde{I}_e^{'}$, which is given by the equation

$$\tilde{I}_e^{'} = -\sum_{j=1}^{n} \tilde{I}_j^{'}$$

The procedure is illustrated with as example involving a three-phase line.

Example E3.3: Consider the three phase line of Example E3.2 illustrated in Figure E3.2. Compute the capacitance matrix of this line by (a) ignoring the earth effect, and (b) taking into account the earth effect. Compare the positive sequence capacitance with and without the earth effect.

Using result (b) compute the capacitive current of this transmission line assuming it is connected to a balanced 115 kV source and the line length is 10 miles.

(a) Solution Ignoring the Earth Effect

Using the formula for the capacitive reactance matrix terms:

$$x'_{ij} = \frac{1}{j\omega 2\pi\varepsilon} \ln\left(\frac{1}{d_{ij}}\right)$$

The capacitive reactance matrix X with d_{ij} expressed in feet is:

$$X = j\begin{bmatrix} 155.15 & -127.70 & -128.69 \\ -127.70 & 155.15 & -104.76 \\ -128.69 & -104.76 & 155.15 \end{bmatrix} \quad M\Omega \bullet m$$

The positive sequence reactance is computed as the difference between the self and the average of the mutual terms, as follows:

$$x_1 = x_s - x_m = \frac{1}{3}(X_{11} + X_{22} + X_{33}) - \frac{1}{3}(X_{12} + X_{13} + X_{23})$$
$$= 155.15 + \frac{1}{3}(127.70{+}128.69{+}104.76) M\Omega\, m = 275.53\, M\Omega\, m$$

and the positive sequence capacitance is obtained from the positive sequence capacitive reactance as:

$$c_1 = \frac{1}{\omega x_1} = \frac{1}{377 \times 275.53 \times 10^6} F/m = 9.6269\, pF/m$$

(b) Solution Including the Earth Effect

Using the formula for the capacitive reactance matrix terms:

$$x'_{ij} = \frac{1}{j\omega 2\pi\varepsilon} \ln\left(\frac{d_{ij'}}{d_{ij}}\right)$$

The first diagonal term is:

$$x'_{11} = \frac{1}{j\omega 2\pi\varepsilon} \ln\left(\frac{d_{11'}}{d_{11}}\right) = \frac{1}{j60 \times 4\pi^2 \times 8.854 \times 10^{-12}} \ln\left(\frac{38.48\, m}{0.0118\, m}\right) \Omega m = 374.72\, M\Omega m$$

Computing the remaining terms in a similar manner, the complete capacitive reactance matrix X is as follows:

$$X = j\begin{bmatrix} 374.72 & 94.17 & 88.95 \\ 94.17 & 378.39 & 114.33 \\ 88.95 & 114.33 & 369.70 \end{bmatrix} \quad M\Omega\, m$$

As in part (a), the positive sequence reactance is computed as the difference between the average self and mutual terms, as follows:

$$x_1 = x_s - x_m = \frac{1}{3}(X_{11} + X_{22} + X_{33}) - \frac{1}{3}(X_{12} + X_{13} + X_{23})$$
$$= \frac{1}{3}(374.72{+}378.39{+}369.70 - 94.17 - 88.95 - 114.33) M\Omega\, m = 275.11\, M\Omega\, m$$

and the positive sequence capacitance is obtained from the positive sequence capacitive reactance as:

$$c_1 = \frac{1}{\omega x_1} = \frac{1}{377 \times 275.11 \times 10^6} F/m = 9.6413\, pF/m$$

Note that as expected, the actual positive sequence capacitance is slightly higher than the value computed ignoring the earth effect. (The error in this case is only 0.15%, and generally decreases with line height).

In order to compute the charging current, we evaluate the capacitive susceptance matrix ***B** = jω**C*** as the inverse of ***X***:

$$B = j\omega C = j\begin{bmatrix} 2.938 & -0.5710 & -0.5303 \\ -0.5710 & 3.0621 & -0.7985 \\ -0.5303 & -0.7985 & 3.0794 \end{bmatrix} \quad nS/m$$

and therefore:

$$C = \begin{bmatrix} 7.7933 & -1.5145 & -1.4067 \\ -1.5145 & 8.0271 & -2.1180 \\ -1.4067 & -2.1180 & 8.1685 \end{bmatrix} \quad pF/m$$

and finally the capacitive current is computed by multiplying the susceptance matrix by the voltage vector corresponding to a balanced 115 kV line-to-line system, as follows:

$$I = j\omega CV = jBV = jB\begin{bmatrix} 66{,}390 \,/\, 0^0 \\ 66{,}390 \,/\, -120^0 \\ 66{,}390 \,/\, 120^0 \end{bmatrix} = \begin{bmatrix} 3.7268 \,/\, 90.578^0 \\ 3.9696 \,/\, -26.961^0 \\ 4.0069 \,/\, -153.550^0 \end{bmatrix} A$$

Note that even though the voltage is balanced the charging current is unbalanced, due to the asymmetry of the line capacitance.

3.2.4 Line Models for Sinusoidal Steady State

We consider the sinusoidal steady-state operating conditions. In this case the imposed voltages and currents on the transmission line vary sinusoidally with frequency f. Since the line is a linear system, the currents and voltages at any point, y, in the line will vary sinusoidally with time. Thus, in general,

$$i(y,t) = \mathrm{Re}\left[\sqrt{2}\tilde{I}(y)e^{j\omega t}\right] \tag{3.32a}$$

$$v(y,t) = \mathrm{Re}\left[\sqrt{2}\tilde{V}(y)e^{j\omega t}\right] \tag{3.32b}$$

where $\tilde{I}(y), \tilde{V}(y)$ are complex numbers (phasors) and $\omega = 2\pi f$. The models of a single- or three-phase line under the conditions described are developed in the next sections.

3.2.4.1: Single-Phase Transmission Line

Let r, L and C be the resistance, inductance and capacitance all per unit length of a single phase line. If we consider an infinitesimal length *dy* of this line, then the lumped parameter model of the infinitesimal length is shown in Figure 3.20.

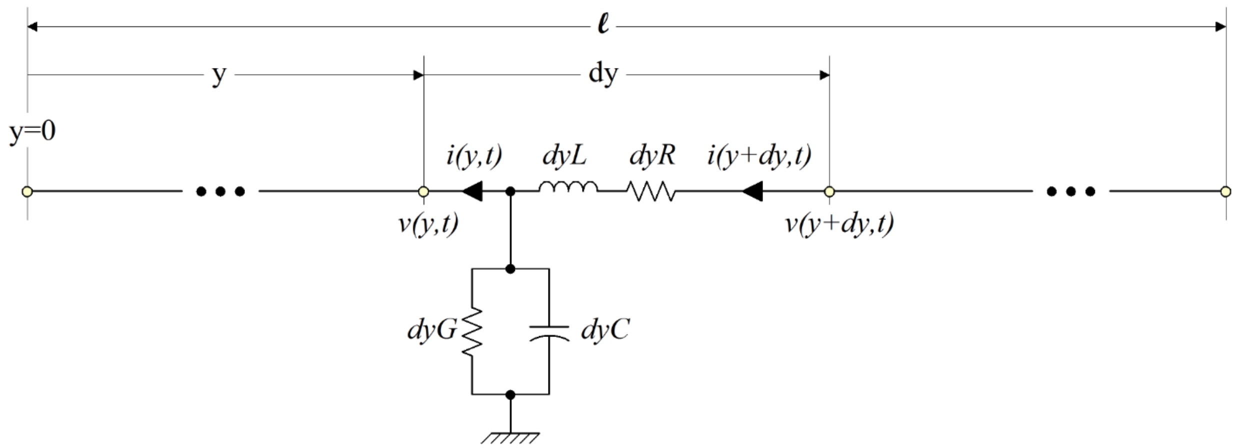


Figure 3.20: Transmission Line with Distributed Parameters

Applying Kirchoff's voltage and current law to this circuit:

$$v(y+dy,t)-v(y,t)=dyRi(y+dy,t)+dyL\frac{di(y+dy,t)}{dt}$$
$$i(y+dy,t)-i(y,t)=dyGv(y,t)+dy\,\mathrm{C}\frac{dv(y,t)}{dt}$$

Dividing both equations by *dy* and taking the limit as *dy* goes to zero, one obtains:

$$\frac{\partial v(y,t)}{\partial y}=Ri(y,t)+L\frac{\partial i(y,t)}{\partial t}$$
$$\frac{\partial i(y,t)}{\partial y}=Gv(y,t)+C\frac{\partial v(y,t)}{\partial t}$$

Differentiation of the first equation with respect to y, yields:

$$\frac{\partial^2 v(y,t)}{\partial y^2}=R\frac{\partial i(y,t)}{\partial y}+L\frac{\partial^2 i(y,t)}{\partial y\partial t}=RGv(y,t)+RC\frac{\partial v(y,t)}{\partial t}+L\frac{\partial^2 i(y,t)}{\partial y\partial t}$$

Differentiation of the second equation with respect to t and substituting the result in above equation one obtains:

$$\frac{\partial^2 v(y,t)}{\partial y^2} = RGv(y,t) + RC\frac{\partial v(y,t)}{\partial t} + L\left(G\frac{\partial v(y,t)}{\partial t} + C\frac{\partial^2 v(y,t)}{\partial t^2}\right)$$
$$= RGv(y,t) + (RC+LG)\frac{\partial v(y,t)}{\partial t} + LC\frac{\partial^2 v(y,t)}{\partial t^2}$$

In summary, the differential equations of a single phase line are:

$$\frac{\partial^2 v(y,t)}{\partial y^2} = RGv(y,t) + (RC+LG)\frac{\partial v(y,t)}{\partial t} + LC\frac{\partial^2 v(y,t)}{\partial t^2}$$
$$\frac{\partial v(y,t)}{\partial y} = Ri(y,t) + L\frac{\partial i(y,t)}{\partial t}$$

Upon substitution of Eqs. (3.32) into above equations, we obtain

$$\sqrt{2}\,\mathrm{Re}\left\{e^{j\omega t}\frac{d^2\tilde{V}(y)}{dy^2}\right\} = \sqrt{2}\,\mathrm{Re}\left\{\,[-\omega^2 LC\tilde{V}(y) + j\omega(GL+CR)\tilde{V}(y) + GR\tilde{V}(y)]e^{j\omega t}\,\right\}$$

$$\sqrt{2}\,\mathrm{Re}\left\{e^{j\omega t}\frac{d\tilde{V}(y)}{dy}\right\} = \sqrt{2}\,\mathrm{Re}\left\{\,e^{j\omega t}(R+j\omega L)\tilde{I}(y)\right\}$$

The equation above must be satisfied for any time t. Thus the coefficients of the time functions on the two sides of the equation must be identical, yielding.

$$\frac{d^2\tilde{V}(y)}{dy^2} = \left[-\omega^2 LC + j\omega(GL+CR) + GR\right]\tilde{V}(y)$$

$$\frac{d\tilde{V}(y)}{dy} = (R+j\omega L)\tilde{I}(y)$$

Upon factorization of the right-hand-side expression, we have

$$\frac{d^2\tilde{V}(y)}{dy^2} = (R+j\omega L)(G+j\omega C)\tilde{V}(y) \qquad (3.33a)$$
$$\frac{d\tilde{V}(y)}{dy} = (R+j\omega L)\tilde{I}(y) \qquad (3.33b)$$

Now let's define

$z_s = \dfrac{1}{y_s} = R + j\omega L$ = series impedance per unit length of the line at frequency ω

$y_{sh} = \dfrac{1}{z_{sh}} = G + j\omega C$ = shunt admittance per unit length of the line at frequency ω

With the new notation, Equation (3.33) become

$$\frac{d^2\tilde{V}(y)}{dy^2} = z_s y_{sh} \tilde{V}(y) \qquad (3.34a)$$

$$\frac{d\tilde{V}(y)}{dy} = z_s \tilde{I}(y) \qquad (3.34b)$$

Equations (3.34) represent the single-phase line model at sinusoidal steady state. The general solution of Eq. (3.34a) is

$$\tilde{V}(y) = ae^{py} + be^{-py} \qquad (3.35)$$

where a, b are constants dependent on the boundary conditions of the line, and

$$p = \sqrt{z_s y_{sh}} = \sqrt{-\omega^2 LC + j\omega(GL + RC) + GR} \qquad (3.36)$$

Note that p is dependent on the angular frequency. The dimensions of the constant p are the inverse of length. The constant p characterizes the propagation of voltage through the transmission line. For this reason it is called the propagation constant. The real and imaginary parts of the propagation constant will be called the attenuation and phase constant, respectively. That is, p = κ + jη, where κ is the attenuation constant and η is the phase constant.

The general solution for the electric current phasor $\tilde{I}(y)$ is obtained by substituting Eq. (3.35) into Eq. (3.34b). The result is

$$\tilde{I}(y) = \frac{p}{z_s}(ae^{py} - be^{-py})$$

Observe that

$$\frac{p}{z_s} = \sqrt{\frac{y_{sh}}{z_s}}$$

Define

$$Z_0 = \frac{1}{Y_0} = \sqrt{\frac{z_s}{y_{sh}}} \tag{3.37}$$

Note that the quantity Z_0 has dimensions of impedance and it is characteristic of the transmission line under consideration. It will be called the characteristic impedance of the line. In terms of the characteristic impedance Z_0, the equation for the line current becomes

$$\tilde{I}(y) = \frac{a}{Z_0} e^{py} - \frac{b}{Z_0} e^{-py} = Y_0 a e^{py} - Y_0 b e^{-py} \tag{3.38}$$

In summary, the general solution for the voltage and current phasors at a location y of a single-phase line is given by Equations (3.35) and (3.38). The solution is expressed in terms of the propagation constant p, the characteristic impedance Z_0, and two constants a and b. The quantities p and Z_0 depend on the parameters of the line, while the constants a, b are dependent on the boundary conditions. If enough boundary conditions are given, for example the terminal voltage and current at an end of the line, the constants a and b can be expressed as a function of the boundary data.

As an example, we will assume that the voltage and current at the receiving end of the line of Figure 3.1 are known to be $\tilde{V}_R$ and $\tilde{I}_R$. Note that the receiving end of this line is characterized with y=0. Then

$$\tilde{V}(y=0) = \tilde{V}_R = a + b$$

$$\tilde{I}(y=0) = \tilde{I}_R = \frac{a}{Z_0} - \frac{b}{Z_0}$$

Upon solution of two equations above for the constants a and b we obtain

$$a = \frac{\tilde{V}_R + Z_0 \tilde{I}_R}{2}$$

$$b = \frac{\tilde{V}_R - Z_0 \tilde{I}_R}{2}$$

Substitution into Equations (3.8) and (3.9) gives us

$$\tilde{V}(y) = \tilde{V}_R \frac{e^{py} + e^{-py}}{2} + Z_0 \tilde{I}_R \frac{e^{py} - e^{-py}}{2} = \tilde{V}_R \cosh(py) + Z_0 \tilde{I}_R \sinh(py) \tag{3.39a}$$

$$\tilde{I}(y) = \frac{\tilde{V}_R}{Z_0} \frac{e^{py} - e^{-py}}{2} + \tilde{I}_R \frac{e^{py} + e^{-py}}{2} = Y_0 \tilde{V}_R \sinh(py) + \tilde{I}_R \cosh(py) \tag{3.39b}$$

Equations (3.39) provide the voltage and current phasors at any location y along the line in terms of the voltage and current at the receiving end of the line (y = 0). Of special interest are the voltage and current at the other end of the line (y = ℓ):

$$\tilde{V}_S = \tilde{V}\left(y=\ell\right) = \tilde{V}_R \cosh(p\ell) + Z_0 \tilde{I}_R \sinh(p\ell)$$
$$\tilde{I}_S = \tilde{I}(y=\ell) = Y_0 \tilde{V}_R \sinh(p\ell) + \tilde{I}_R \cosh(p\ell)$$

In compact matrix notation:

$$\begin{bmatrix} \tilde{V}_S \\ \tilde{I}_S \end{bmatrix} = \begin{bmatrix} \cosh(p\ell) & Z_0 \sinh(p\ell) \\ Y_0 \sinh(p\ell) & \cosh(p\ell) \end{bmatrix} \begin{bmatrix} \tilde{V}_R \\ \tilde{I}_R \end{bmatrix}$$

This equation states that sending-end voltage and current are a linear combination of the receiving-end voltage and current, and vice versa. Three parameters describe this model completely: (a) the characteristic impedance of the line Z_0; (b) the propagation constant of the line, p; and (c) the length of the line, ℓ. Note that the model depends only on the product $p\ell$ and the characteristic impedance Z_0. Alternatively, the following parameters completely describe the single-phase transmission line: (a) $\mathrm{A} = \cosh\left(p\ell\right)$, (b) $B = Z_0 \sinh\left(p\ell\right)$, and (c) $C = Y_0 \sinh\left(p\ell\right)$. In terms of the parameters A, B, C, the line equations (3.39) become

$$\tilde{V}_S = A\tilde{V}_R + B\tilde{I}_R \tag{3.40a}$$
$$\tilde{I}_S = C\tilde{V}_R + A\tilde{I}_R \tag{3.40b}$$

These parameters are known as the A, B, C constant of the line. Note that

$$A^2 - BC = \cosh^2\left(p\ell\right) - \sinh^2\left(p\ell\right) = 1.0$$

Thus the parameters A, B, and C are not independent. Knowledge of the two is enough to determine the third.

The above model of a single phase transmission lines can be represented with an equivalent circuit. The derivation of equivalent circuits is discussed in section 3.2.5.

3.2.4.2 Three-Phase Transmission Line

The same analysis can be applied to three-phase transmission lines. Assuming sinusoidal steady state, Equations (3.30a) and (3.31b) of the three-phase transmission line become

$$\frac{d^2\tilde{V}(y)}{dy^2} = (R + j\omega L)(G + j\omega C)\tilde{V}(y) \tag{3.41a}$$

$$\frac{d\tilde{V}(y)}{dy} = (R + j\omega L)\tilde{I}(y) \tag{3.41b}$$

Define the following matrices:

$$Z_s = R + j\omega L$$
$$Y_{sh} = G + j\omega C$$

Then

$$\frac{d^2\tilde{V}(y)}{dy^2} = Z_s Y_{sh}\tilde{V}(y) \tag{3.42a}$$
$$\frac{d\tilde{V}(y)}{dy} = Z_s\tilde{I}(y) \tag{3.42b}$$

The foregoing matrix differential equations in complex variables fully describe the performance of a general three-phase transmission line. Solution of these equations for specified boundary conditions will yield the electric currents and voltages of any phase at any location of the line. However, solution of the equations above is rather difficult. In the following section we discuss transformations that decompose the matrix equations (3.42) into scalar equations. In this way, the solution of the matrix equations (3.42) reduces to the solution of a set of scalar equations.

3.2.4.3 Modal Decomposition

The model of a three-phase transmission line under sinusoidal steady state condition is defined by Equations (3.42). Solution of these equations is in general complex because the matrices Z_s, Y_{sh} are full matrices resulting in a set of three coupled differential equations. To simplify the solution, observe that it is possible to find a transformation T of the voltage and current vector $\tilde{V}(y)$ and $\tilde{I}(y)$ as follows:

$$\tilde{V}(y) = T\tilde{V}^m(y) \quad \text{or} \quad \tilde{V}^m(y) = T^{-1}\tilde{V}(y) \tag{3.43a}$$
$$\tilde{I}(y) = T\tilde{I}^m(y) \quad \text{or} \quad \tilde{I}^m(y) = T^{-1}\tilde{I}(y) \tag{3.43b}$$

where T is a 3×3 matrix, $\tilde{V}^m(y)$ are the transformed voltages at location y of the line, and $\tilde{I}^m(y)$ are the transformed currents at location y of the line. Substitution of the transformation above into Equation (3.42) and subsequent pre-multiplication of the resulting equation by T results in

$$\frac{d^2\tilde{V}^{(m)}(y)}{dy^2} = T^{-1}Z_sY_{sh}T\tilde{V}^{(m)}(y) = T^{-1}Z_sTT^{-1}Y_{sh}T\tilde{V}^{(m)}(y) \qquad (3.44a)$$

$$\frac{d\tilde{V}^{(m)}(y)}{dy} = T^{-1}Z_sT\tilde{I}^{(m)}(y) \qquad (3.44b)$$

Now assume that T has been selected in such a way that the matrices $T^{-1}Z_sT$ and $T^{-1}Y_{sh}T$ are diagonal matrices. In this case Equations (3.44) represent six uncoupled differential equations. The voltages $\tilde{V}^m(y)$ are called the modal voltages of the line and the transformation T is called a modal transformation matrix. Similarly, the currents $\tilde{I}^m(y)$ are called the modal currents. The procedures is called the modal decomposition. The advantages of modal decomposition are obvious. Solution of the decoupled equations (3.44) is identical to solution methods for single phase lines.

3.2.4.4 Sequence Models

A special case of the modal decomposition results in what is known as the sequence models of a three-phase line. Specifically, many transmission lines are transposed or their construction is such that the mutual parameters (inductance, capacitance) are approximately the same for any pair of phases and the phase self-parameters are also approximately the same for the three phases. For this reason it is justifiable to approximate a three-phase power line with a symmetric line. Mathematically, this is equivalent to assuming that the matrices Z and Y' have the following symmetric structure:

$$Z_s = \begin{bmatrix} z_{s,s} & z_{s,m} & z_{s,m} \\ z_{s,m} & z_{s,s} & z_{s,m} \\ z_{s,m} & z_{s,m} & z_{s,s} \end{bmatrix}$$

$$Y_{sh} = \begin{bmatrix} y_{sh,s} & y_{sh,m} & y_{sh,m} \\ y_{sh,m} & y_{sh,s} & y_{sh,m} \\ y_{sh,m} & y_{sh,m} & y_{sh,s} \end{bmatrix}$$

Note that if the matrices $Z_s \text{ and } Y_{sh}$ do not have this form, which is usually the case, they are put in this form using the following equations:

$$z_{s,s} = \frac{1}{3}\left(z_{aa} + z_{bb} + z_{cc}\right)$$

$$z_{s,\mathrm{m}} = \frac{1}{3}\left(z_{ab} + z_{bc} + z_{ca}\right)$$

$$y_{sh,s} = \frac{1}{3}\left(y_{aa} + y_{bb} + y_{cc}\right)$$

$$y_{sh,\mathrm{m}} = \frac{1}{3}\left(y_{ab} + y_{bc} + y_{ca}\right)$$

The product $Z_s Y_{sh}$ of the two matrices is computed to be

$$Z_s Y_{sh} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_2 \\ \alpha_2 & \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_2 & \alpha_1 \end{bmatrix}$$

where

$$\alpha_1 = z_{s,s} y_{sh,s} + 2 z_{s,m} y_{sh,m}$$

$$\alpha_2 = z_{s,m} y_{sh,m} + z_{s,s} y_{sh,m} + z_{s,m} y_{sh,s}$$

Now under the discussed assumption of symmetry, the modal transformation matrix T is defined as follows:

$T = \begin{bmatrix} 1 & 1 & 1 \\ a^2 & a & 1 \\ a & a^2 & 1 \end{bmatrix}$, where $a = e^{j120^0}$. Note that the inverse of this matrix is:

$$T^{-1} = \frac{1}{3}\begin{bmatrix} 1 & a & a^2 \\ 1 & a^2 & a \\ 1 & 1 & 1 \end{bmatrix}$$

The modal voltages $\tilde{V}^m(y)$ in this case will be denoted by

$$\tilde{V}_{120}(y) = \begin{bmatrix} \tilde{V}_1(y) \\ \tilde{V}_2(y) \\ \tilde{V}_0(y) \end{bmatrix}$$

and the modal currents will be denoted by

$$\tilde{I}_{120}(y) = \begin{bmatrix} \tilde{I}_1(y) \\ \tilde{I}_2(y) \\ \tilde{I}_0(y) \end{bmatrix}$$

Upon substitution of this modal transformation into Equation (3.44), we obtain

$$\frac{d^2\tilde{V}_{120}(y)}{dy^2} = M_{seq}\tilde{V}_{120}(y) \tag{3.45a}$$

$$\frac{d\tilde{V}_{120}(y)}{dy} = Z_{seq}\tilde{I}_{120}(y) \tag{3.45b}$$

where

$$M_{seq} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_0 \end{bmatrix}$$

$$Z_{seq} = \begin{bmatrix} z_1 & 0 & 0 \\ 0 & z_1 & 0 \\ 0 & 0 & z_0 \end{bmatrix}$$

$$m_1 = \alpha_1 - \alpha_2 = p_1^2$$
$$m_0 = \alpha_1 + 2\alpha_2 = p_0^2$$
$$z_1 = z_s - z_m$$
$$z_0 = z_s + 2z_m$$

The matrix equations (3.45) represent six scalar equations. It is expedient to write the six scalar equations explicitly and grouped them into three groups of two as follows:

$$\frac{d^2\tilde{V}_1(y)}{dy^2} = p_1^2\tilde{V}_1(y) \tag{3.46a}$$

$$\frac{d\tilde{V}_1(y)}{dy} = z_1\tilde{I}_1(y) \tag{3.46b}$$

$$\frac{d^2\tilde{V}_2(y)}{dy^2} = p_1^2\tilde{V}_2(y) \tag{3.47a}$$

$$\frac{d\tilde{V}_2(y)}{dy} = z_1\tilde{I}_2(y) \tag{3.47b}$$

$$\frac{d^2\tilde{V}_0(y)}{dy^2} = p_0^2\tilde{V}_0(y) \tag{3.48a}$$

$$\frac{d\tilde{V}_0(y)}{dy} = z_0\tilde{I}_0(y) \tag{3.48b}$$

It is now apparent that Equations (3.46), (3.47) and (3.48) represent three single-phase lines. We shall refer to Equation (3.46) as the positive sequence model of the line, Equations (3.47) as the negative sequence model, and Equations (3.48) as the zero sequence model of the line. Note that the parameters (p_1, z_1) of the negative sequence model are identical to those of the positive sequence model. Collectively, we shall refer to Equation (3.45) or equivalently. Equations (3.46), (3.47), and (3.48) as the sequence model of a three-phase line. The modal voltages and currents will be referred to as the symmetrical components of the currents and voltages. In addition, the parameters of the sequence models are defined as follows:

p_1, p_0 will be called the positive (or negative) and zero sequence propagation constants of the line.

z_1, z_0 will be called the per-unit length positive (or negative) and zero sequence series impedance of the line.

For the purpose of completing the discussion of the sequence model, recall that

$$M_{seq} = T^{-1}Z_sY_{sh}T$$

Consider the following:

$$M_{seq} = T^{-1}Z_sY_{sh}T = T^{-1}Z_sTT^{-1}Y_{sh}T = Z_{s,seq}Y_{sh,seq}$$

where:

$$Z_{s,seq} = T^{-1}Z_sT \qquad\qquad Y_{sh,seq} = T^{-1}Y_{sh}T$$

Upon evaluation of $Y_{sh,seq}$, we have

$$Y_{sh,seq} = \begin{bmatrix} y_{sh,1} & 0 & 0 \\ 0 & y_{sh,1} & 0 \\ 0 & 0 & y_{sh,0} \end{bmatrix}$$

where

$$y_{sh,1} = y_{sh,s} - y_{sh,m}$$

$$y_{sh,0} = y_{sh,s} + 2y_{sh,m}$$

Note that y'_1, y'_0 are the per-unit length positive (or negative) and zero sequence shunt admittance of the line.

In terms of the parameters y'_1, y'_0, the propagation constants p_1, p_2 and p_0 are

$$p_1 = p_2 = \sqrt{z_{s,1} y_{sh,1}} \tag{3.49a}$$

$$p_0 = \sqrt{z_{s,0} y_{sh,0}} \tag{3.49b}$$

and the characteristic impedances:

$$Z_0^1 = Z_0^2 = \sqrt{\frac{z_{s,1}}{y_{sh,1}}} \tag{3.50a}$$

$$Z_0^0 = \sqrt{\frac{z_{s,0}}{y_{sh,0}}} \tag{3.50b}$$

In summary, application of the symmetrical component transformation on the three-phase line equations results in the sequence models of the line (i.e., the positive, negative, and zero sequence models). Each model is identical to a single-phase line model. The parameters of the positive sequence models are equal to the parameters of the negative sequence model.

A physical interpretation of the sequence models of a three-phase transmission line is expedient. For this purpose, assume that only one symmetrical component of the voltage or current is present. As an example, assume that only the positive sequence current is present,

i.e. $\tilde{I}_1(y) \neq 0, \quad \tilde{I}_2(y) = 0, \quad and \quad \tilde{I}_0(y) = 0$

The actual phase currents $\tilde{I}_a(y)$, $\tilde{I}_b(y)$, $\tilde{I}_c(y)$ are obtained from the inverse transformation T^{-1}:

$$\tilde{I}_{abc}(y) = T \begin{bmatrix} \tilde{I}_1(y) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{I}_1(y) \\ \tilde{I}_1(y) e^{-j120^\circ} \\ \tilde{I}_1(y) e^{-j240^\circ} \end{bmatrix}$$

As is evident from the equation above, the three phase currents are balanced and of the positive sequence. The case is depicted in Figure 3.22a, which illustrates the three phase currents. Note that the electric current in the ground is zero.

Similarly, if we assume that only the negative sequence component is present $\left[i.e. \quad \tilde{I}_1(y) = 0, \quad \tilde{I}_2(y) \neq 0, \quad and \quad \tilde{I}_0(y) = 0 \right]$, the actual phase currents are

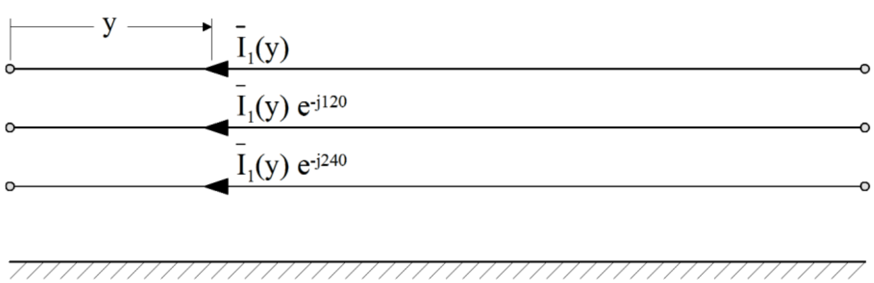
$$\tilde{I}_{abc}(y) = T\begin{bmatrix} 0 \\ \tilde{I}_2(y) \\ 0 \end{bmatrix} = \begin{bmatrix} \tilde{I}_2(y) \\ \tilde{I}_2(y)e^{j120^\circ} \\ \tilde{I}_2(y)e^{j240^\circ} \end{bmatrix}$$

Again, as is evident from equation above, the three phase currents are balanced but of the negative sequence. The case is depicted in Figure 3.22b, which illustrates the three phase currents. Note that the electric current in the earth is zero.

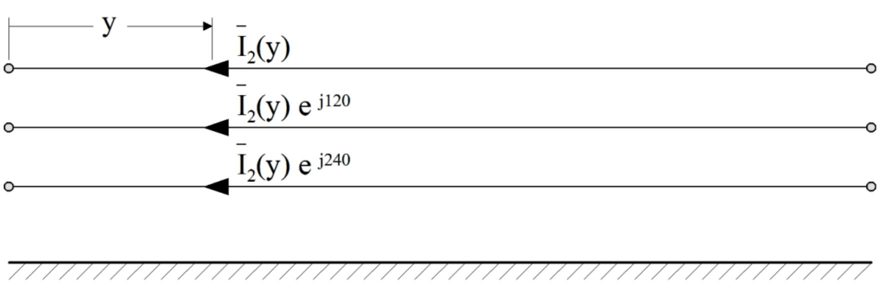
Finally, if we assume that only the zero sequence component is present, $\left[i.e.\ \ \tilde{I}_1(y)=0,\ \ \tilde{I}_2\left(y\right)=0,\ \ and\ \ \tilde{I}_0\left(y\right)\neq 0 \right]$, the actual phase currents are

$$\tilde{I}_{abc}(y) = T\begin{bmatrix} 0 \\ 0 \\ \tilde{I}_0(y) \end{bmatrix} = \begin{bmatrix} \tilde{I}_0(y) \\ \tilde{I}_0(y) \\ \tilde{I}_0(y) \end{bmatrix}$$

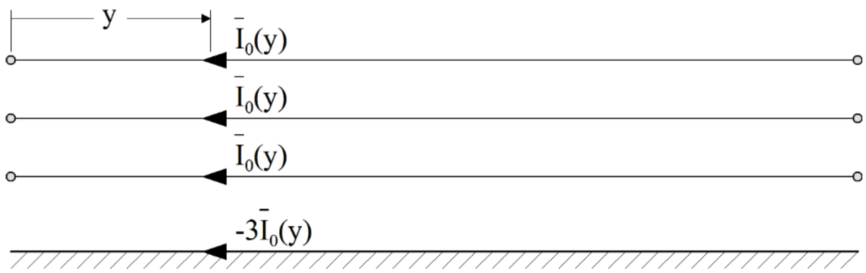
As is evident from the equation above, all three phase currents are identical. Sequence cannot be defined for these currents-thus the name "zero sequence". The earth current will be the negative of the sum of the phase currents [i.e., $-3\tilde{I}_0(y)$]. The case is depicted in Figure 3.22c.



(a)



(b)



(c)

Figure 3.20: Illustration of the Symmetrical Components on a Transmission Line

(a) Positive Sequence Components, (b) Negative Sequence Components,(c) Zero Sequence Components

Example E3.4: Consider the transmission line of Example E3.2 and Example E3.3. Compute the sequence parameters of the line.

Solution: The Z matrix of the line, computed at 60 Hz, and then symmetrized is:

$$Z = \begin{bmatrix} 0.2323 + j0.8901 & 0.0593 + j0.4330 & 0.0593 + j0.4330 \\ 0.0593 + j0.4330 & 0.2323 + j0.8901 & 0.0593 + j0.4330 \\ 0.0593 + j0.4330 & 0.0593 + j0.4330 & 0.2323 + j0.8901 \end{bmatrix} \times 10^{-3}\ ohms/meter$$

The Y' matrix of the line, computed at 60 Hz, and then symmetrized is:

$$Y' = \begin{bmatrix} j3.0331 & -j0.5556 & -j0.5556 \\ -j0.5556 & j3.0331 & -j0.5556 \\ -j0.5556 & -j0.5556 & j3.0331 \end{bmatrix} \times 10^{-9}\ S/m$$

The product ZY' is

$$ZY' = \begin{bmatrix} -2.2186 + jo.6387 & -0.5782 + j0.0178 & -0.5782 + j0.0178 \\ -0.5782 + j0.0178 & -2.2186 + jo.6387 & -0.5782 + j0.0178 \\ -0.5782 + j0.0178 & -0.5782 + j0.0178 & -2.2186 + jo.6387 \end{bmatrix} \times 10^{-12}\ \mathrm{m}^{-2}$$

Upon application of the transformation T, we have

$$z_1 = (0.173 + j0.4571) \times 10^{-3}\ \Omega/\ m$$

$$z_0 = (0.3509 + j1.7561) \times 10^{-3}\ \Omega/\ m$$

$$y'_1 = j3.5887 \times 10^{-9}\ S/\ m$$

$$y'_0 = j1.9219 \times 10^{-9}\ S/\ m$$

$$m_1 = (-1.6404 + j0.6209) \times 10^{-12}\ m^{-2}$$

$$m_0 = (-3.375 + j0.6743) \times 10^{-12}\ m^{-2}$$

The characteristic impedance and propagation constants of the sequence components are:

$$Z_0^1 = \sqrt{\frac{z_1}{y'_1}} = 369 e^{-j10.365^\circ}\ \Omega$$

$$p_1 = p_2 = \sqrt{z_1 y'_1} = 1.3244 \times 10^{-6} e^{j79.63^\circ}\ \mathrm{m}^{-1}$$

Zero sequence components:

$$Z_0^0 = \sqrt{\frac{z_0}{y'_0}} = 965 e^{-j5.65^\circ} \quad ohms$$

$$p_0 = \sqrt{z_0 y'_0} = 1.8552 \times 10^{-6} e^{j84.35^\circ} \ \mathrm{m}^{-1}$$

In summary of this section, the symmetrical component transformation provides a tool for the simplified solution of the equations of a multiphase line. It also yields the sequence models of a three-phase line. In this case the analysis of three-phase lines is reduced to the analysis of three single-phase transmission lines, the positive sequence, negative sequence and zero sequence line models.

3.2.5 Transmission Line Equivalent Circuits

In previous section we have developed models of single-phase, as well as three-phase lines, under steady-state conditions. The models are in terms of the A, B, C parameters or alternatively, in terms of the characteristic impedance, propagation constant, and line length. An alternative representation of the transmission lines under steady-state conditions is by means of equivalent circuits. This approach is more attractive because of the familiarity of engineers with circuits. This section presents the computation of equivalent circuits from the transmission line parameters. Only the single-phase line case will be demonstrated. Since a three-phase line can be reduced to three single-phase lines by means of the symmetrical component transformation, the extension to three-phase lines will be left to the reader as an exercise.

Consider Equations (3.40) of a single-phase line in terms of the terminal currents and voltages. From realization theory it is known that a two-port circuit can be found which is described with the same equations. Furthermore, this two-port circuit is not unique. From the multiplicity of equivalent circuits, one particular circuit has been popular among power engineers: the π equivalent. This circuit is introduced next.

To a transmission line with constants A, B, C, corresponds a π-equivalent circuit with elements Y_π, Y'_π as in Figure 3.21. The elements Y_π, Y'_π of the π-equivalent circuit are computed by first expressing the line terminal currents as a function of the line terminal voltages and subsequent application of circuit theory. Specifically, the line terminal currents as a function of line terminal voltages in terms of the parameters A, B and C, are:

$$\tilde{I}_1 = \frac{A}{B}\tilde{V}_1 - \frac{1}{B}\tilde{V}_2 \qquad (3.51a)$$

$$\tilde{I}_2 = -\frac{1}{B}\tilde{V}_1 + \frac{A}{B}\tilde{V}_2 \qquad (3.51b)$$

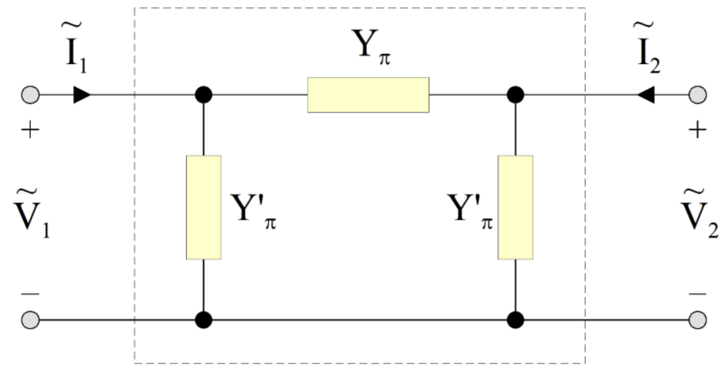


Figure 3.21: π -equivalent Circuit

On the other hand, the equation for the circuit of Figure 3.23 are:

$$\tilde{I}_1 = (Y_\pi + Y_\pi^{'})\tilde{V}_1 - Y_\pi \tilde{V}_2$$
$$\tilde{I}_1 = -Y_\pi \tilde{V}_1 + (Y_\pi^{'} + Y_\pi)\tilde{V}_2$$

For equivalence, the following must hold:

$$Y_\pi = \frac{1}{B} \tag{3.52a}$$

$$Y_\pi^{'} = \frac{A-1}{B} \tag{3.52b}$$

Equations (3.52) define the parameters of the π-equivalent circuit of a line. These parameters are the series admittance of the equivalent circuit, Y_π, and the shunt admittance of the equivalent circuit, Y'_π. The impedance parameters of the π equivalent circuit will be:

$$Z_\pi = \frac{1}{Y_\pi} = B = Z_0 \sinh(p\ell)$$

$$Z_\pi^{'} = \frac{1}{Y_\pi^{'}} = \frac{B}{A-1} = \frac{Z_0 \sinh(p\ell)}{\cosh(p\ell) - 1}$$

Nominal π-Equivalent Circuit: The nominal π-equivalent circuit of a transmission line is an approximation of the exact equivalents. In general, these approximations are valid for short lines; thus the name "short-line equivalent" is alternatively used. Consider the π-equivalent circuit described by the parameters Z_π, and Z'_π, as derived earlier. The nominal π-equivalent circuit is obtained by approximating the hyperbolic sine and cosine functions. Specifically, assuming that $p\ell \ll 1$, (this assumption is equivalent to the assumption of short line, i.e. ℓ is small), the functions are expanded into a series and then only the major terms are retained:

$$\sinh(p\ell) \cong p\ell$$

$$\cosh(p\ell) \cong 1 + \frac{(p\ell)^2}{2}$$

Substitution of the approximations above in the equations for the parameters Z_π, and Z'_π, yields:

$$Z_\pi \cong Z_0 p\ell = z\ell = (r + j\omega L)\ell \qquad (3.53a)$$

$$Z'_\pi \cong \frac{Z_0 p\ell}{1 + \frac{(p\ell)^2}{2} - 1} = 2\frac{z'}{\ell} = \frac{2}{(g + j\omega C)\ell} \qquad (3.53b)$$

Normally, the nominal pi-equivalent approximation can be made when the product pl is small. For example for 0.1% accuracy, it should be less than 0.025. For 60 Hz model, this means a line of about less than 12 miles.

The computation of equivalent circuits for a transmission line will be illustrated with an example.

Example E3.5: Consider the positive sequence model of the three-phase line of Example E3.4. The computed parameters are $Z_0 = 369e^{-j10.36^\circ}$ ohms and $p = 1.3244\times10^{-6}\, e^{j79.63^\circ}\, m^{-1}$. Assume that the line is 85 miles long and compute:

(a) The π-equivalent circuit.
(b) The nominal π-equivalent circuit.
(c) Compare the two models

<u>Solution</u>: First, the A, B, C parameters of the line are computed as follows:

$$p\ell = 0.18113e^{j79.63^\circ}$$

$$\cosh p\ell = 0.984659 + j0.005809 = 0.984676e^{j0.34^\circ}$$

$$\sinh p\ell = 0.032092 + j0.177323 = 0.180204e^{j79.74^\circ}$$

$$A = 0.984676e^{j0.34^\circ},\ B = 66.4953e^{j69.38^\circ},\ C = 0.000488e^{j90.1^\circ}$$

(a)

$$Z_\pi = B = (23.417 + j62.235)\ \Omega$$

$$Z'_\pi = \frac{B}{A-1} = (8.489 - j4053.59)\ \Omega$$

The π-equivalent circuit is illustrated in Fig. E6.1a.

(b) $$Z_\pi = z\ell = Z_0 p\ell = 66.837e^{j69.27^\circ}\ \Omega = (23.658 + j62.509)\ \Omega$$

$$Z'_\pi = \frac{2z'}{\ell} = \frac{2Z_0}{p\ell} = 4079.42e^{-j90^\circ}\ \Omega = (-j4074.4)\ \Omega$$

The nominal π-equivalent circuit is illustrated in Figure E3.6c.

(c) The two equivalent circuits are very close.

Example E3.6: A three phase transmission line has the following per unit length positive sequence parameters

Resistance: R = 0.08 ohms/mile
Inductance: L = 1.1×10^{-6} Henries/meter
Capacitance: C = 10.8×10^{-12} Farads/meter

The line is 200 miles long.

a) Compute the positive sequence π-equivalent circuit of the line.
b) Compute the positive sequence nominal π equivalent circuit of the line.

Solution:

a)
$$z = 0.08 + j0.667\ \Omega/\text{mi}$$
$$y = j6.551\times10^{-6}\ \text{S/mi}$$

$$Z_c = \sqrt{\frac{z}{y}} = 320.28\angle -3.418^\circ = 319.71 - j19.10\Omega$$

$$\gamma\ell = \ell\sqrt{zy} = 0.4196\angle 86.58^\circ = 0.0250 + j0.4189$$

$$Z' = Z_c \sinh \gamma\ell = 130.417\angle 83.36^\circ\ \Omega = 15.08 + j\ 129.64\ \Omega$$

$$\frac{y'}{2} = \frac{1}{Z_c}\tanh\left(\frac{\gamma\ell}{2}\right) = 1.19\times10^{-6} + j6.646\times10^{-4}\ S$$

$$\frac{2}{y'} = 2.682 - j\ 1504.2\ \Omega$$

b) The nominal equivalent circuit parameters are:

$$Z = zl = 16 + j133.4\ \ \Omega = 134.35\ \underline{/\ 83.16^0}\ \ \Omega$$

$$\frac{2}{Y} = \frac{2}{\ell y} = 1526.5\,\Omega$$

The results for this example are shown in Figure E3.9.

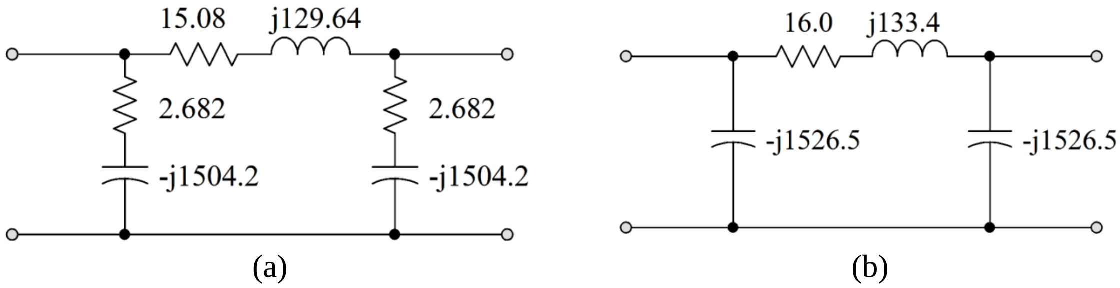


Figure E3.6: The Positive Sequence Equivalent Circuits
(a) pi-equivalent Circuit, (b) Nominal pi-equivalent Circuit
(All indicated values are in ohms)

The above computational methods for the parameters of lines and equivalent circuits of lines have been demonstrated for the fundamental power frequency. The same procedures can be applied for any frequency. As an example we apply these computational procedures for the line parameters and equivalent circuits for the 7^{th} harmonic (420 Hz).

Example E3.6: Consider the 230 kV transmission line of Figure E3.6. For simplicity, assume that the phase conductors are aluminum one inch diameter of conductivity 40,000,000 S/m. The line is 57 miles long. Compute the positive, negative and zero sequence π-equivalent circuit of the line for the 7^{th} harmonic. For simplicity, neglect the shield wires.

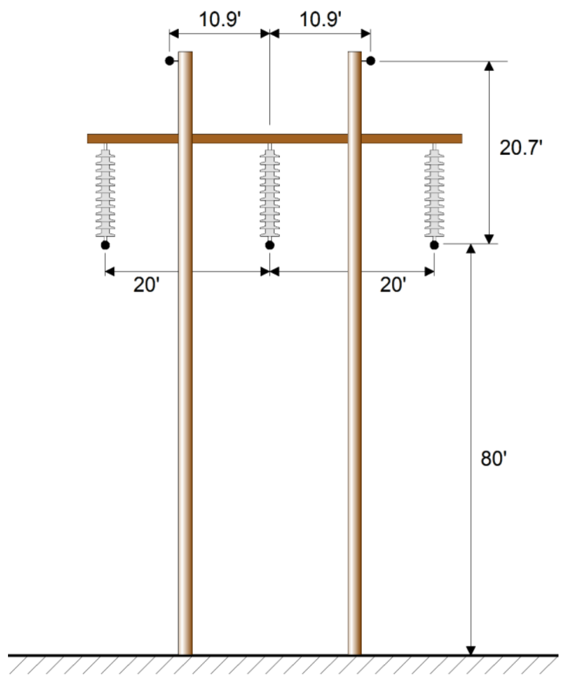


Figure E3.6

Solution: The resistance and inductance matrices are computed using Carsons equations. The results are:

$$k = \sqrt{k\omega\sigma} = 364.26\, m^{-1}$$

ka=4.625

$$M_0(ka) = 4.7$$
$$\theta_0(ka) = 162.0^0$$
$$M_1(ka) = 4.3$$
$$\theta_1(ka) = 260.0^0$$

$$r_{ac} = 9.96x10^{-5}\ ohms/m$$

$$\xi = 0.5689$$

$$d = ae^{-\frac{\xi}{4}} = 0.011016m = 0.03614\ ft$$

$$r_e = 0.00159f = 0.6678 ohms/mi = 41.5x10^{-5}\ ohms/m$$

$$D_e = 2160\sqrt{\frac{\rho}{f}} = 1{,}053.97\ ft = 321.25\, m$$

$$R = 10^{-5}\begin{bmatrix} 51.46 & 41.5 & 41.5 \\ 41.5 & 51.46 & 41.5 \\ 41.5 & 41.5 & 51.46 \end{bmatrix}\Omega / m$$

$$L = 10^{-6}\begin{bmatrix} 2.056 & 0.793 & 0.654 \\ 0.793 & 2.056 & 0.793 \\ 0.654 & 0.793 & 2.056 \end{bmatrix} H / m$$

$$C = 2\pi\varepsilon\begin{bmatrix} 8.253 & 2.087 & 1.416 \\ 2.087 & 8.253 & 2.087 \\ 1.416 & 2.087 & 8.253 \end{bmatrix}^{-1} = 2\pi\varepsilon\begin{bmatrix} 0.1312 & -0.0294 & -0.0151 \\ -0.0294 & 0.1360 & -0.0294 \\ -0.0151 & -0.0294 & 0.1312 \end{bmatrix}$$

From above matrices the following parameters are obtained:

Positive/Negative sequence:

$$r_1 = 99.6\,\mu\Omega / m, \quad L_1 = 1.3093\,\mu H / m, \quad C_1 = 8.758\,pF / m$$

Zero sequence:

$$r_0 = 1344.6\mu\Omega / m, \; L_0 = 3.5493\,\mu H / m, \quad C_0 = 4.651\,pF / m$$

The parameters of the equivalent circuits are:

Positive/Negative Sequence

3.3 Cable Modeling

Power cables are very common for medium and low voltage distribution systems. Recently, we have seen increased used of UG transmission cables 138 kV to 345 kV. There is a variety of cable designs. Figure 3.22 illustrates a concentric neutral medium voltage cable construction. Figure 3.23 illustrates a three phase cable construction (medium or high voltage). Figure 3.24 illustrates a three wire secondary voltage cable (2x120V).

Cable designs have rather complicated geometries and accurate analysis and computation of their parameters is very complex and for all practical purposes it is done by computer modeling. In this chapter, we present the basis of and computational procedures for the evaluation of the parameters of cables. The theory is followed by examples of cable parameters for usual cable geometries.

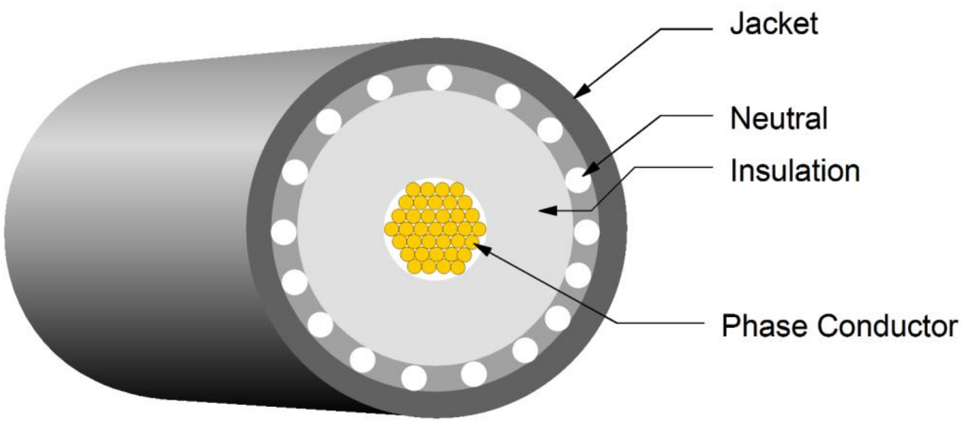


Figure 3.22: Concentric Neutral Medium Voltage Cable

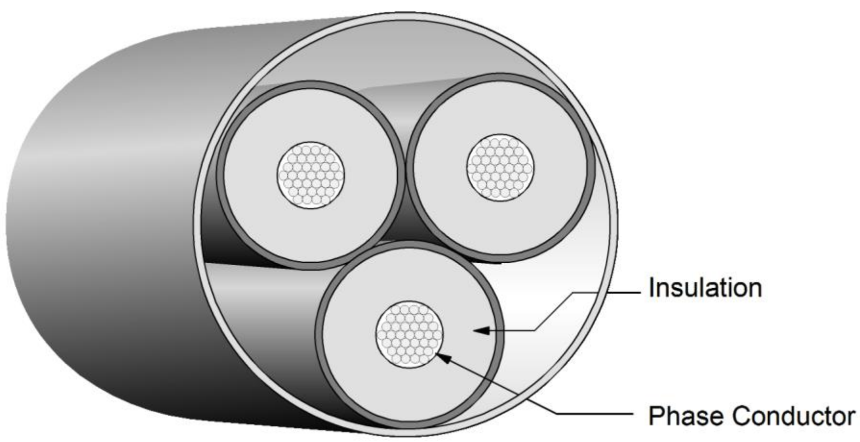


Figure 3.23: Three-Phase Medium Voltage Cable

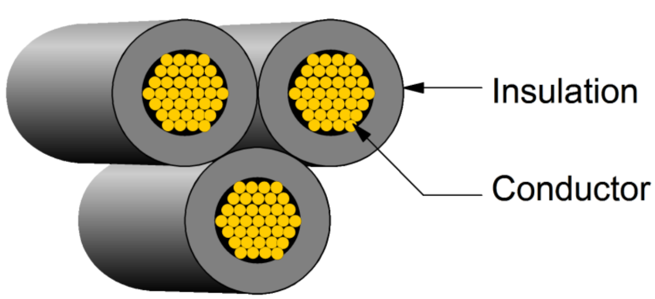


Figure 3.24: Secondary System 600V Class Cable

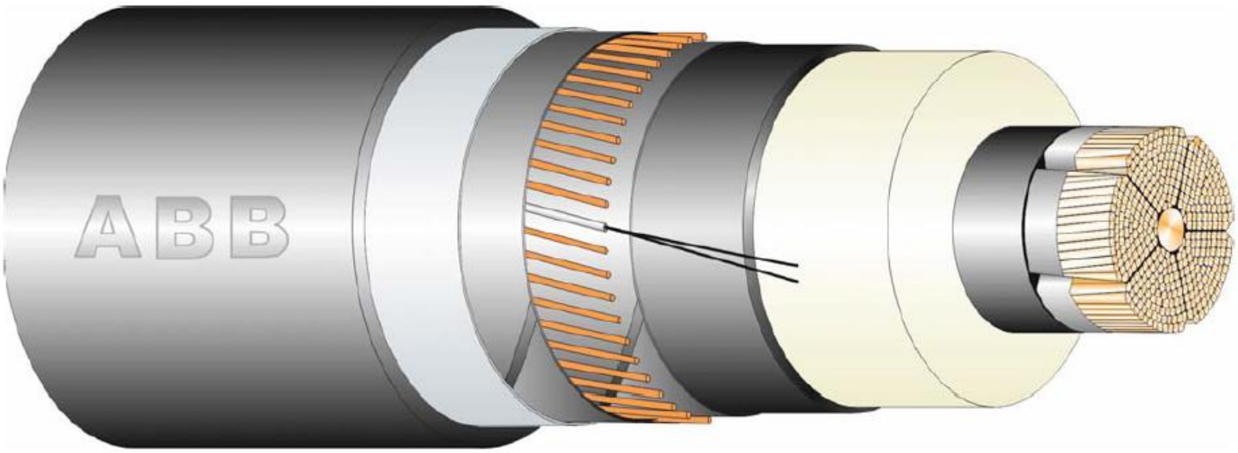


Figure 3.25: 135 kV, 3000 kcm Transmission Cable – Manufacturer: ABB
(note the fibers in a metal tube occupying the location of one copper shield wire)

3.3.1 Methodologies for Cable Parameter Computation

to be added

3.3.2 Typical Cable Parameters

The figures that follow present typical variations of concentric neutral cable parameters versus frequency. One should observe that while the cable reactance is rather insensitive to soil resistivity, the cable resistance is quite sensitive to soil resistivity, especially as the frequency increases. For all practical purposes, the parameters of cables are computed by computer programs.

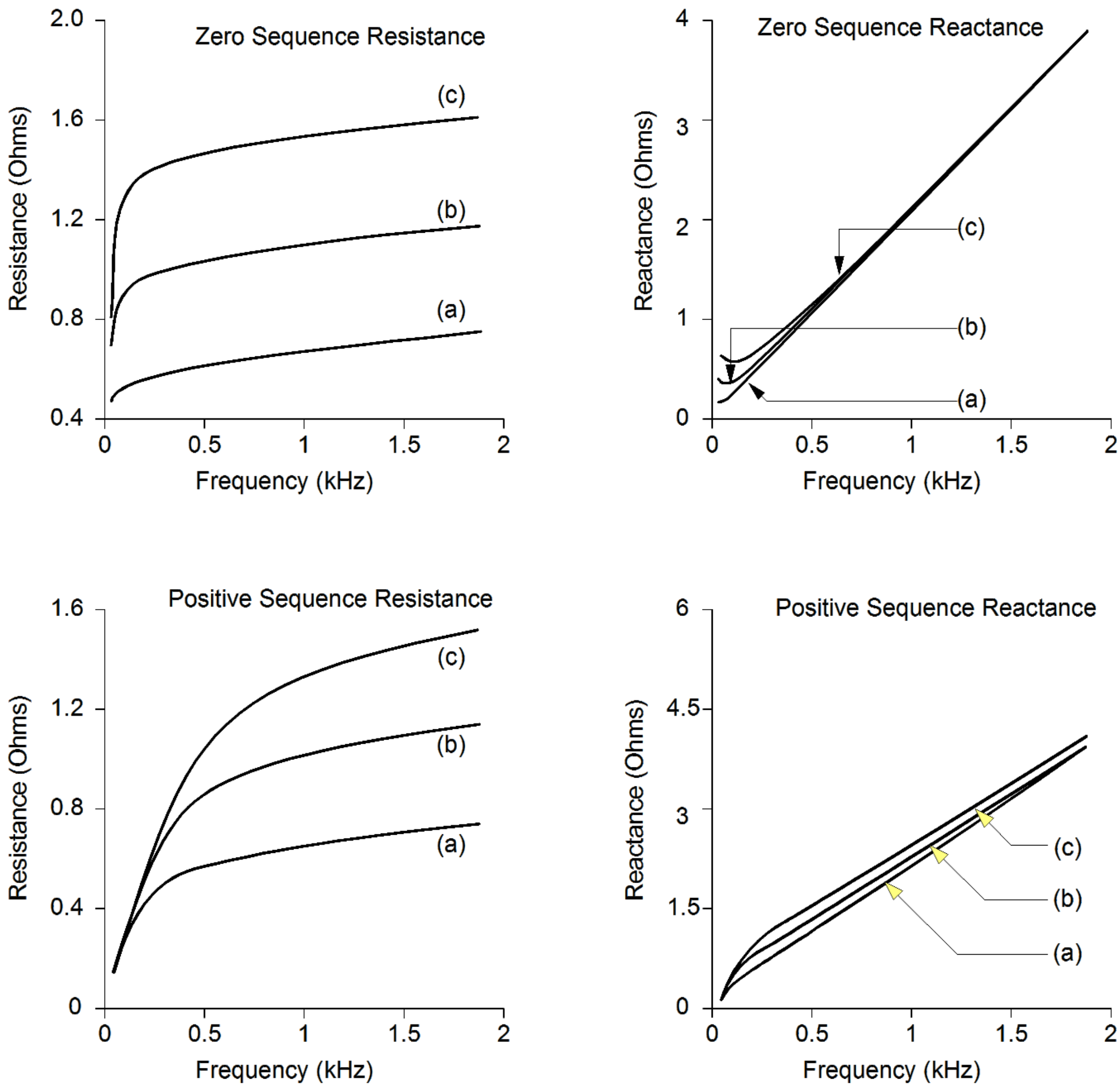


Figure 3.25: Parametric Analysis of 15 kV Concentric Neutral Cable Sequence Parameters
[(a) Soil Resistivity of 10 ohm.m, (b) Soil Resistivity of 100 ohm.m, (c) Soil Resistivity of 1000 ohm.m]

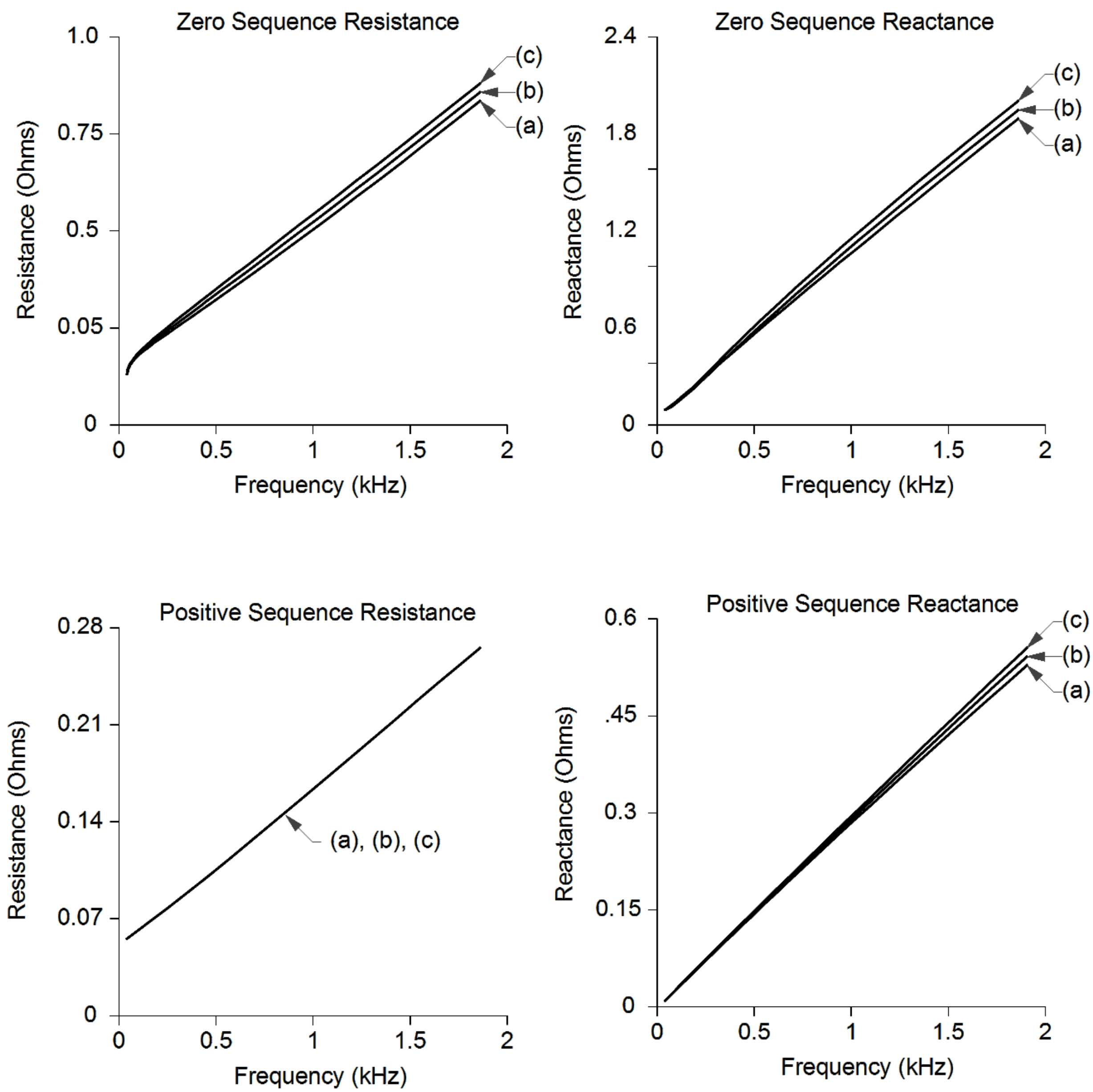
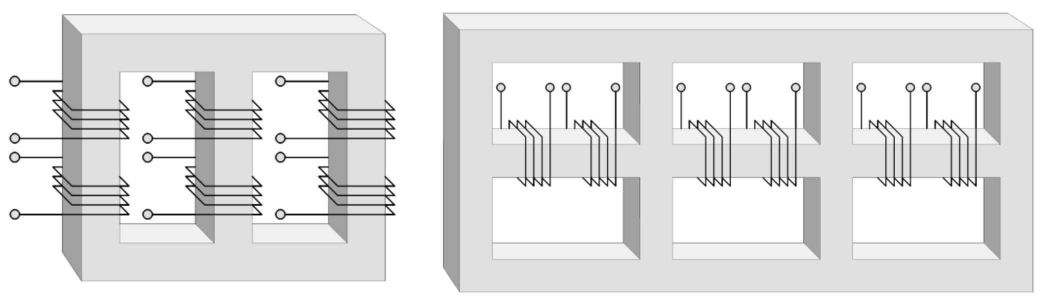


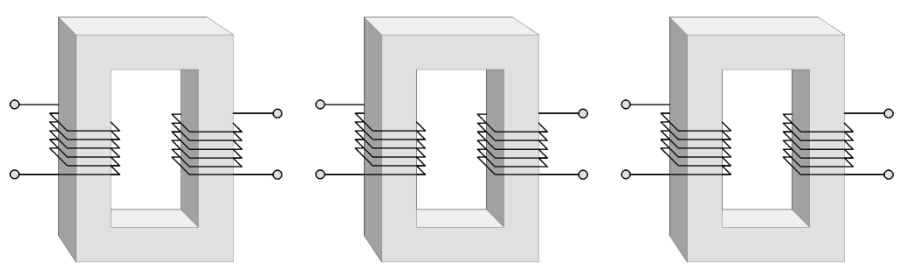
Figure 3.26: Parametric Analysis of 600 V Cable Sequence Parameters
[(a) Soil Resistivity of 10 ohm.m, (b) Soil Resistivity of 100 ohm.m, (c) Soil Resistivity of 1000 ohm.m]

3.4 Transformer Modeling

Transformers can be single phase or three-phase, two windings or multiple windings, and some windings may be center-tapped. In general, the coils of a transformer are electrically isolated from each other enabling the isolation of the circuits that may be connected to these coils. Three phase transformers can be constructed in a number of ways. Three of the most usual constructions are illustrated in Figure 3.27. Figure 3.27a illustrates a three phase core type transformer. The core has three legs, on each leg there are two windings for a total of six windings. Similarly, Figure 3.27b illustrates a shell type transformer which also has six windings. Figure 3.27c illustrates a "bank" of three single phase transformers. This arrangement also has six windings. The six windings of any configuration (a), (b), or (c) are grouped in two groups of three, the primary and the secondary. For example, in Figure 3.27a the primary may be the three windings located on the upper part of each leg and the secondary may be the other three winding. Both the primary and secondary windings may be connected in a delta or wye configurations leading to four possible arrangements of a three phase transformer: (a) delta-delta, (b) wye-delta, (c) delta-wye and (d) wye-wye. These arrangements are schematically represented in Figure 3.28. Note that from the circuit point of view, all three phase transformer constructions are similar, i.e. all have six winding grouped into three phases. However, the magnetic circuit of each one of these constructions is different. For example, the three phase transformer bank consists of three independent magnetic circuits. The shell and core type three phase transformers are characterized with coupled magnetic circuits of the three phases.



(a) (b)



(c)

Figure 3.27: Three Phase Transformers
(a) Core Type, (b) Shell Type, (c) Three Single Phase Transformer Bank

The model of a three phase transformer bank is the simplest since it consists of the interconnection of three single phase transformers. Replacing each one of the single phase transformers with its equivalent circuit, the equivalent circuit of the three phase transformer is obtained. This has been done in Figure 3.29 where the simplified equivalent circuit of a single phase transformer has been used. The Figure illustrates a delta-wye connection.

In subsequent paragraphs we will consider first the ideal three phase transformer model for the purpose of examining its basic characteristics. Then the non-ideal transformer model will be studied. The use of the symmetrical transformation to the three phase transformer model will result in the sequence models.

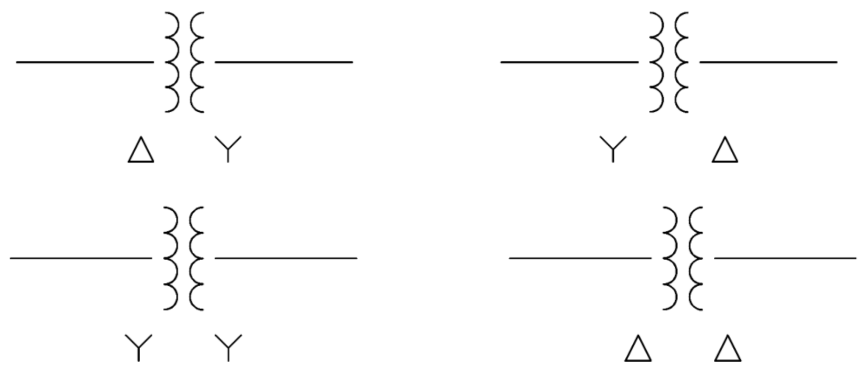


Figure 3.28: Schematic Representation of Three Phase Transformers

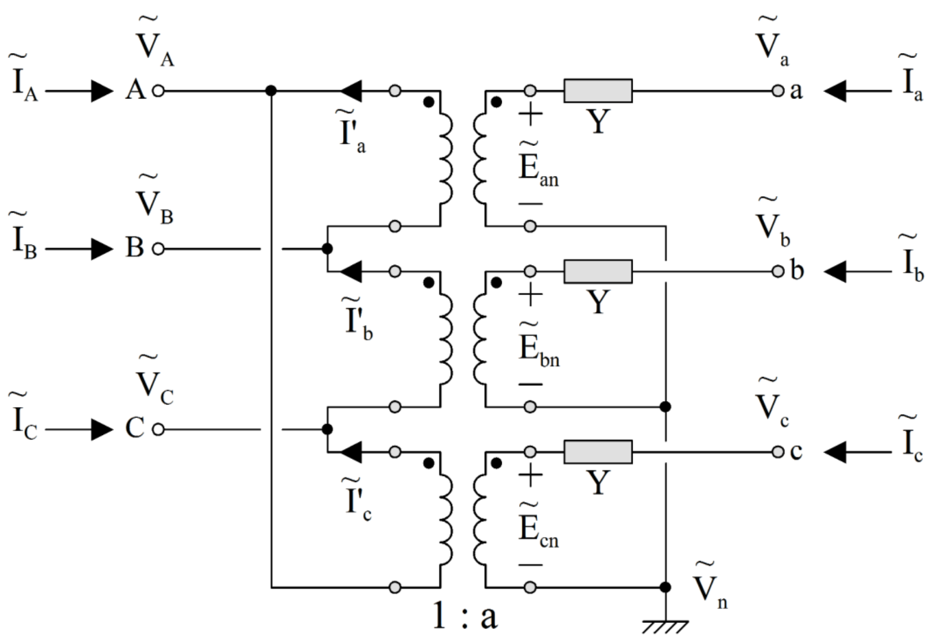


Figure 3.29: Delta-Wye Connected Three Phase Transformer Model

3.4.1 The Ideal Three Phase Transformer

An ideal three phase transformer consists of three ideal single phase transformers. The transformer of Figure 3.29 will be ideal if $Y = \infty$ (short circuit). The voltage relationships of an ideal three phase transformer are:

$$\tilde{V}_{AB} = a^{-1}\tilde{V}_{an}$$
$$\tilde{V}_{BC} = a^{-1}\tilde{V}_{bn}$$
$$\tilde{V}_{CA} = a^{-1}\tilde{V}_{cn}$$

Under balanced operating conditions, the voltages will be:

$$\tilde{V}_{Bn} = \tilde{V}_{An}e^{-j120^o}$$
$$\tilde{V}_{Cn} = \tilde{V}_{An}e^{-j240^o}$$
$$\tilde{V}_{bn} = \tilde{V}_{an}e^{-j120^o}$$
$$\tilde{V}_{cn} = \tilde{V}_{an}e^{-j240^o}$$

Note that:

$$\tilde{V}_{AB} = \tilde{V}_{An} - \tilde{V}_{Bn} = \tilde{V}_{An} - \tilde{V}_{An}e^{-j120^o} = \sqrt{3}\tilde{V}_{An}e^{j30^o}$$
$$\tilde{V}_{BC} = \tilde{V}_{Bn} - \tilde{V}_{Cn} = \tilde{V}_{An}e^{-j120^o} - \tilde{V}_{An}e^{+j120^o} = \sqrt{3}\tilde{V}_{An}e^{-j90^o} = \sqrt{3}\tilde{V}_{Bn}e^{j30^o}$$
$$\tilde{V}_{CA} = \tilde{V}_{Cn} - \tilde{V}_{An} = \tilde{V}_{An}e^{+j120^o} - \tilde{V}_{An}e^{-j120^o} = \sqrt{3}\tilde{V}_{An}e^{-j210^o}$$

Now the relationship between the primary and secondary voltages can be found.

$$\tilde{V}_{An} = \frac{e^{-j30^0}}{a\sqrt{3}}\tilde{V}_{an}$$
$$\tilde{V}_{Bn} = \frac{e^{-j30^0}}{a\sqrt{3}}\tilde{V}_{bn}$$
$$\tilde{V}_{Cn} = \frac{e^{-j30^0}}{a\sqrt{3}}\tilde{V}_{cn}$$

Above equations indicate that the per phase (positive sequence) equivalent model of a delta-wye connected ideal three phase transformer is a single phase ideal transformer with transformation ratio $n = a\sqrt{3}e^{j30^0}$.